A Hamiltonian Approach to Equations of Economics

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SDEA2014

Joint work with R Naz and A Chaudhry

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Outline of presentation

Motivation and history
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- Aims and scope
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- Mathematical Formulation: Hamiltonian viewpoint
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- Two widely used Economic models:
  - The Ramsey model with a constant relative risk aversion (CRRA) utility function with Cobb Douglas technology
  - One-sector AK model of endogenous growth
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  - The Ramsey model with a constant relative risk aversion (CRRA) utility function with Cobb Douglas technology
  - One-sector AK model of endogenous growth
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- Optimal control theory and dynamic optimization
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Optimal control theory and dynamic optimization

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Neoclassical economic growth models (Ramsey 1928 and Lucas 1988), optimal firm-level investment (Eisner and Strotz 1963), human capital and earnings (Ben-Porath 1967)
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- Optimal control theory and dynamic optimization

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- Neoclassical economic growth models (Ramsey 1928 and Lucas 1988), optimal firm-level investment (Eisner and Strotz 1963), human capital and earnings (Ben-Porath 1967)

- A dynamical system of ordinary differential equations obtained for control, state and costate variables
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There have been various approaches to deal with dynamic economic models arising from current value Hamiltonian system, both qualitative and quantitative (see Cass and Shell 1976, Ruiz-Tamarit and Ventura-Marco 2011).

Most of these models were solved using specific restrictions (like Mehlam’s 2005 assumption of Leontiff technology), numerical solutions (like Mulligan and Sala-i-Martin (1991)) or linear approximations around steady states (Barro and Sala-i-Martin (2004)).
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The critical problem is that for the underlying nonlinear dynamical system of economics there is a lack of general solution procedure not only for higher order systems but for one state and costate variables as well.

It is true to say that nonlinear dynamical systems evade closed-form solutions in general. However, the lack of a general procedure inhibits the search for reductions and solutions of such type of nonlinear equations even when solutions do exist.
There are some well-known closed-form solutions that appear in the literature (see, e.g. Ruiz-Tamarit (2008); Chilarescu (2008, 2009); Hiraguchi (2009); Guerrini (2010); Diele et al. (2011)). These solutions have been obtained by seemingly disparate approaches.
There are some well-known closed-form solutions that appear in the literature (see, e.g. Ruiz-Tamarit (2008); Chilarescu (2008, 2009); Hiraguchi (2009); Guerrini (2010); Diele et al. (2011)). These solutions have been obtained by seemingly disparate approaches.

Independent of the knowledge of explicit solutions, dynamic local stability of certain systems (see Rodriguez (2004); Brock and Scheinkman (1976); Rodriguez (1996)) can be characterized by qualitative or numerical approaches.
Aims and scope

- Development of a new approach which yields reductions and closed-form solutions for dynamical systems arising from current value Hamiltonian of Economics
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- Development of a new approach which yields reductions and closed-form solutions for dynamical systems arising from current value Hamiltonian of Economics

- Introduction of Hamiltonian framework for several control, state and costate variables
Hamiltonian formulation for Economics

Mathematical formulas: a partial Hamiltonian approach
Maximum principle for current value Hamiltonian
A Hamiltonian viewpoint for equations of Economics

In economics, the maximum principle consists of first-order ordinary differential equations in what are called state and costate variables. Also the further requirement is that the current-value Hamiltonian be maximized with respect to the control variables.

Suppose that one maximizes

$$\int_0^T H_1(t, q, u) e^{-rt} dt$$  \hspace{1cm} (1)$$

subject to

$$\dot{q} = H_2(t, q, u),$$  \hspace{1cm} (2)$$

where the integrand contains the discount factor $e^{-rt}$, $t$ is the time, $q$ is the state variable and $u$ the control variable.
Then the current-value Hamiltonian is

\[ H(t, q, u, p) = H_1(t, q, u) + p H_2(t, q, u) \]  \hspace{1cm} (3)

in which \( p \) is referred to as the costate variable. The necessary conditions for optimal control (see Chiang 1992) are \( \partial H / \partial u = 0 \) as well as

\[ \dot{q} = \frac{\partial H}{\partial p}, \]
\[ \dot{p} = -\frac{\partial H}{\partial q} + rp. \]  \hspace{1cm} (4)

The system is not in the form of the canonical Hamiltonian equations due to nonzero term \( rp \). Reduction techniques developed in the classical literature DO NOT HOLD.
Let $t$ be the independent variable and 
$(q, p) = (q^1, \ldots, q^n, p_1, \ldots, p_n)$ the phase space coordinates. The derivatives of $q^i$, $p_i$ with respect to $t$ are

$$
\dot{q}_i = D(q_i), \quad \dot{p}_i = D(p_i), \quad i = 1, 2, \cdots, n,
$$

where

$$
D = \frac{\partial}{\partial t} + \dot{q}_i \frac{\partial}{\partial q_i} + \dot{p}_i \frac{\partial}{\partial p_i} + \cdots,
$$

is the total derivative operator with respect to $t$. The summation convention is utilized for repeated indices. The variables $t, q, p$ are independent and connected only by the differential relations (5).
The Euler operator

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - D \frac{\partial}{\partial q^i}, \ i = 1, 2, \cdots, n$$ \hspace{1cm} (7)

and the variational operator

$$\frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i} - D \frac{\partial}{\partial p_i}, \ i = 1, 2, \cdots, n.$$ \hspace{1cm} (8)
The action of the operators (7) and (8) on

\[ L(t, q, \dot{q}) = p_i \dot{q}^i - H(t, q, p) \]  \hspace{1cm} (9)

equated to zero yields the present value Hamilton equations

\[ \dot{q}^i = \frac{\partial H}{\partial p_i}, \]

\[ \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad i = 1, \ldots, n. \]  \hspace{1cm} (10)

That is \( \frac{\delta L}{\delta q^i} = 0 \) and \( \frac{\delta L}{\delta p_i} = 0 \) results in (10). Equation (9) is the well-known Legendre transformation which relates the Hamiltonian and Lagrangian, where \( \dot{q}^i = \partial L / \partial p_i, \)

\[ \dot{p}_i = -\partial H / \partial q^i. \]
Symmetries of a present value Hamiltonian
Generators of point symmetries in the space \((t, q, p)\) are operators of the form

\[ X = \xi(t, q, p) \frac{\partial}{\partial t} + \eta^i(t, q, p) \frac{\partial}{\partial q^i} + \zeta_i(t, q, p) \frac{\partial}{\partial p_i}. \]  

(11)

The operator as in (11) is a generator of point symmetry of the Hamiltonian system (10) if

\[ \dot{\eta}^i - \dot{q}^i \dot{\xi} - X\left( \frac{\partial H}{\partial p_i} \right) = 0, \]

\[ \dot{\zeta}_i - \dot{p}_i \dot{\xi} + X\left( \frac{\partial H}{\partial q^i} \right) = 0, \quad i = 1, \ldots, n \]  

(12)

on the system (10).
The current value Hamiltonian satisfies

\[ \dot{q}^i = \frac{\partial H}{\partial p_i}, \]

\[ \dot{p}^i = -\frac{\partial H}{\partial q^i} + \Gamma_i, \quad i = 1, 2, \ldots, n. \]

where \( \Gamma_i \) is a nonzero function of time \( t \).
Symmetries of current value Hamiltonian
The operator \( X \)

\[
X = \xi(t, q, p) \frac{\partial}{\partial t} + \eta^i(t, q, p) \frac{\partial}{\partial q^i} + \zeta_i(t, q, p) \frac{\partial}{\partial p_i}. \tag{14}
\]

is a generator of point symmetry of the current value Hamiltonian system (13) if

\[
\dot{\eta}^i - \dot{q}^i \dot{\xi} - X \left( \frac{\partial H}{\partial p_i} \right) = 0,
\]

\[
\dot{\zeta}_i - \dot{p}_i \dot{\xi} + X \left( \frac{\partial H}{\partial q^i} - \Gamma_i \right) = 0, \quad i = 1, \ldots, n \tag{15}
\]
on the system (13). Note that (15) is evidently different from (12) due to the nonzero term \( \Gamma_i \).
Partial Hamiltonian Operators
An operator \( X \) is a partial Hamiltonian operator corresponding to a current value Hamiltonian as in (13), if there exists a function \( B(t, q, p) \) such that

\[
\zeta_i \frac{\partial H}{\partial \eta^i} + p_i D(\eta^i) - X(H) - HD(\xi) = D(B) + (\eta^i - \xi \frac{\partial H}{\partial p_i})(-\Gamma_i).
\]

(16)

Note that if \( H \) is a present value Hamiltonian, then equation (16) becomes the usual determining equation for symmetries of the Hamiltonian action since \( \Gamma_i = 0 \).
First integrals
The non-canonical Hamiltonian system (13) has the first integral

\[ I = p_i \eta^i - \xi H - B \]  \hspace{1cm} (17)

for some gauge function \( B = B(t, q, p) \) if and only if the operator \( X \)

\[ X = \xi(t, q, p) \frac{\partial}{\partial t} + \eta^i(t, q, p) \frac{\partial}{\partial q^i} + \zeta_i(t, q, p) \frac{\partial}{\partial p_i} \]  \hspace{1cm} (18)

satisfies the determining equation (16).
A simple illustrative example
Consider the following mathematical example:

Maximize

$$
\int_0^\infty [\alpha q - \beta q^2 - \alpha u^2 - \gamma u] e^{-rt} dt
$$

subject to

$$\dot{q} = u,$$

where $\alpha, \beta, \gamma$ are all positive, $r$ is a discount factor, $q(t)$ is the state variable and $u(t)$ is the control variable.
Hamiltonian function and maximum principal:

The current value Hamiltonian function is defined as

\[ H(t, q, p, u) = \alpha q - \beta q^2 - \alpha u^2 - \gamma u + pu \]  

where \( p(t) \) is called the costate variable. The necessary first order conditions for optimal control are:

\[ \frac{\partial H}{\partial u} = 0 \]  

\[ \dot{q} = \frac{\partial H}{\partial p} \]  

\[ \dot{p} = -\frac{\partial H}{\partial q} + rp \]
Equation (22)-(24) with $H$ given by (21) yields

$$p = 2\alpha u + \gamma$$  \hspace{1cm} (25)$$

$$\dot{q} = u$$  \hspace{1cm} (26)$$

$$\dot{p} = 2\beta q - \alpha + pr$$  \hspace{1cm} (27)$$

Equations (25)-(27) need to be solved for $p(t), q(t), u(t)$. Of course the direct easy way for this problem is to eliminate $p$, $u$ by utilizing (25)-(27) in order to obtain a scalar linear second order ordinary differential equation in $q$ which is amenable to straightforward integration. We explain here how we can find the solution by using the partial Hamiltonian approach introduced above.
Determination of partial Hamiltonian operators:
The partial Hamiltonian operator determining equation from (16) is

\[
\zeta \frac{\partial H}{\partial p} + p D(\eta) - X(H) - HD(\xi) = D(B) + (\eta - \xi \frac{\partial H}{\partial p})(-\Gamma). \quad (28)
\]

Expansion of equation (28) yields

\[
p(\eta_t + \dot{q}\eta_q) - \eta(\alpha - 2\beta q) - (\alpha q - \beta q^2)
- \alpha u^2 - \gamma u + pu)(\xi_t + \dot{q}\xi_q)
= B_t + \dot{q}B_q + (\eta - \xi u)(-rp), \quad (29)
\]

in which we assume that \(\xi = \xi(t, q), \eta = \eta(t, q), B = B(t, q)\).
Equation (29) with the help of (25)-(26) can be written as

\[
(2\alpha u + \gamma)(\eta_t + u\eta_q) - (\xi_t + u\xi_q)(\alpha q - \beta q^2 - \alpha u^2 - \gamma u + 2\alpha u^2 + \gamma u) \\
-\eta(\alpha - 2\beta q) = B_t + uB_q + (\eta - \xi u)(-2r\alpha u - \gamma r). 
\]
Separating equation (30) with respect to powers of \( u \) as \( \xi, \eta, B \) do not contain \( u \), we have

\[
\begin{align*}
\text{(31)} & : u^3 : -\alpha \xi_q = 0, \\
\text{(32)} & : u^2 : 2\alpha \eta_q - \alpha \xi_t = 2\alpha r \xi, \\
\text{(33)} & : u : 2\alpha \eta_t + \gamma \eta_q = B_q - 2\alpha r \eta + \gamma r \xi, \\
\text{(34)} & : u^0 : \gamma \eta_t - \eta(\alpha - 2\beta q) - \xi_t(\alpha q - \beta q^2) = B_t - r \gamma \eta.
\end{align*}
\]

System (31)-(33) yields

\[
\begin{align*}
\xi &= a(t), \quad \eta = (\frac{1}{2} \dot{a} + ra)q + b(t), \\
B &= \alpha (\frac{1}{2} \ddot{a} + r \dot{a})q^2 + \alpha r (\frac{1}{2} \dot{a} + ra)q^2 + 2\alpha b q + 2\alpha rbq + \frac{1}{2} \gamma \dot{a}q + d(t).
\end{align*}
\]
Substituting $\xi, \eta, B$ from (35) in (34) and then separating w.r.t powers of $q$ we have

\[ q^2 : \frac{1}{2} \alpha \ddot{a} + \frac{3}{2} \alpha r \dot{a} + (\alpha r - 2\beta) \dot{a} - 2\beta r a = 0, \tag{36} \]

\[ q : \frac{3}{2} (r \gamma - \alpha) \dot{a} + r (r \gamma - \alpha) a + 2b \beta = 2\alpha \ddot{b} + 2\alpha r \dot{b}, \tag{37} \]

\[ q^0 : \gamma \dot{b} - \alpha b + \gamma rb = \dot{d}. \tag{38} \]
The solution of equations (36)-(38) for $a, b, d$ with general 
$\alpha, \beta, \gamma, r$ is purely formal and depend on the roots of the 
characteristic equation. Clearly there are three lengthy 
solutions for $a$ and two for $b$. To be transparent, we have 
selected values. Therefore we seek solution of equations 
(36)-(38) for $a, b, d$ with specific values $\alpha = \gamma = r = 1$ and 
$\beta = 2$ and we arrive at

$$
a(t) = c_1 e^{-t} + c_2 e^{2t} + c_3 e^{-4t}, \\
b(t) = c_4 e^{t} + c_5 e^{-2t}, \\
d(t) = c_4 e^{t} + c_5 e^{-2t} + c_6,
$$

(39)

where $c_1, \cdots, c_6$ are arbitrary constants.
Finally, we obtain the following $\xi, \eta, B$ after substituting $a, b, d$ from (39) into (35)

$$\begin{align*}
\xi &= c_1 e^{-t} + c_2 e^{2t} + c_3 e^{-4t}, \\
\eta &= (\frac{1}{2} c_1 e^{-t} + 2c_2 e^{2t} - c_3 e^{-4t})q + c_4 e^{t} + c_5 e^{-2t}, \\
B(t) &= (6c_2 e^{2t} + 3c_3 e^{-4t})q^2 + (-\frac{1}{2} c_1 e^{-t} + c_2 e^{2t} - 2c_3 e^{-4t} \\
&\quad + 4c_4 e^{t} - 2c_5 e^{-2t})q + c_4 e^{t} + c_5 e^{-2t} + c_6.
\end{align*}$$

(40)
The generators $X_i$ form a vector space. By choosing one of the constants as one and the rest as zero in turn we have the following five operators and gauge terms:

\begin{align*}
    X_1 &= e^{-t} \frac{\partial}{\partial t} + \frac{1}{2} q e^{-t} \frac{\partial}{\partial q}, \quad B_1 = -\frac{1}{2} e^{-t} q \\
    X_2 &= e^{2t} \frac{\partial}{\partial t} + 2 q e^{2t} \frac{\partial}{\partial q}, \quad B_2 = 6 q^2 e^{2t} + q e^{2t} \\
    X_3 &= e^{-4t} \frac{\partial}{\partial t} - q e^{-4t} \frac{\partial}{\partial q}, \quad B_3 = 3 q^2 e^{-4t} - 2 q e^{-4t} \\
    X_4 &= e^{t} \frac{\partial}{\partial q}, \quad B_4 = 4 q e^{t} + e^{t} \\
    X_5 &= e^{-2t} \frac{\partial}{\partial q}, \quad B_5 = -2 q e^{-2t} + e^{-2t}.
\end{align*}
In general the $X_i$’s are not symmetries of the system

$$\begin{align*}
\dot{q} &= \frac{1}{2}p - \frac{1}{2}, \\
\dot{p} &= 4q + p - 1.
\end{align*}$$

which has the operators (41). For example in the case of $X_4$ we have that the first of equations (15) gives $\zeta = 2e^t$. However, the second equation of (15) is not satisfied as easily can be verified.
Construction of first integrals from partial Hamiltonian operators and gauge term:

Now, first integrals satisfying $\frac{dI}{dt} = 0$ corresponding to operators and gauge terms given in (41) can be computed from (17) and the following integrals result.

\[
I_1 = \left[ \frac{1}{2} pq - (q - 2q^2 - u^2 - u + pu) + \frac{1}{2} \right] e^{-t}, \\
I_2 = \left[ 2pq - (q - 2q^2 - u^2 - u + pu) - 6q^2 - q \right] e^{2t}, \\
I_3 = \left[ -pq - (q - 2q^2 - u^2 - u + pu) - 3q^2 + 2q \right] e^{-4t}, \\
I_4 = [p - 4q - 1] e^t, \\
I_5 = [p + 2q - 1] e^{-2t}.
\]

(43)

There are five first integrals of which two are functionally independent.
**Optimal solution via first integrals:**

Equations (25)-(28) need to be solved for $p(t), q(t), u(t)$ with $\alpha = \gamma = r = 1, \beta = 2$. We demonstrate here how one can find a solution by using first integrals. We derive the solution associated with the first integral $I_4$. As $dI/dt = 0$ and thus $I = constant$, we have

$$[p - 4q - 1]e^t = A_1,$$  \hspace{1cm} (44)

where $A_1$ is an arbitrary constant and this gives

$$p(t) = 4q + 1 + A_1 e^{-t}.$$  \hspace{1cm} (45)

From (25), $u = \frac{p - 1}{2}$ and after using $p$ from (45), we have

$$u(t) = \frac{4q + A_1 e^{-t}}{2}.$$  \hspace{1cm} (46)
Thus if $q(t)$ is known we can get the optimal path $p(t)$ and $u(t)$ from (45) and (46). Equation (20) with $u$ from (46) yields

$$\dot{q} = \frac{4q + A_1 e^{-t}}{2},$$  

(47)

and this is a first order linear equation in $q(t)$. The solution of equation (47) is

$$q(t) = \frac{A_1}{2} e^{-t} + A_2 e^{2t}.$$  

(48)
Ramsey neoclassical model with CRRA utility function
We consider the following Ramsey neoclassical growth model where the representative consumer’s utility maximization problem is defined as

$$\max \int_0^\infty e^{-rt} c^{1-\sigma} dt, \quad \sigma \neq 0, 1$$

subject to capital accumulation equation and parameter restriction

$$\dot{k}(t) = k^\beta - \delta k - c, \quad k(0) = k_0, \quad 0 < \beta < 1$$

where $c(t)$ is the consumption per person, $k(t)$ is the capital labor ratio, $\beta, \delta, r$ are the capital share, depreciation rate, rate of time preferences respectively. The intertemporal elasticity of substitution is given by $1/\sigma$ and $k_0$ is the initial capital stock.
The current value Hamiltonian function for this model is defined as

\[ H(t, c, k, \lambda) = c^{1-\sigma} + \lambda(k^\beta - \delta k - c), \]  

(51)

where \( \lambda(t) \) is the costate variable. The necessary first order conditions for optimal control are

\[ \dot{\lambda} = (1 - \sigma)c^{-\sigma} \]  

(52)

\[ \dot{k} = k^\beta - \delta k - c \]  

(53)

\[ \dot{\lambda} = -\lambda(\beta k^{\beta - 1} - \delta) + \lambda r \]  

(54)
The transversality condition is

$$\lim_{t \to \infty} e^{-rt} \lambda(t) k(t) = 0.$$  \hspace{1cm} (55)

From (52) and (54), the growth rate of consumption is given by

$$\frac{\dot{c}}{c} = \frac{\beta}{\sigma} k^{\beta-1} - \frac{1}{\sigma} (\delta + r).$$  \hspace{1cm} (56)
We seek a solution $\lambda(t), k(t), c(t)$ of equations (52)-(54) by utilizing the Hamiltonian approach. The partial Hamiltonian determining equation (28) for the Hamiltonian (51) yields

\[
\lambda(\eta_t + \dot{k}\eta_k) - \eta\lambda(\beta \dot{k}^{\beta-1} - \delta) - [c^{1-\sigma} \\
+ \lambda(\dot{k}^{\beta} - \delta k - c)](\xi_t + \dot{k}\xi_k) \\
= B_t + \dot{k}B_k + (\eta - \xi \frac{\partial H}{\partial \lambda})(-r\lambda),
\]

in which we assume that $\xi = \xi(t, k), \eta = \eta(t, k), B = B(t, k)$. A Hamiltonian Approach to Equations of Economics – p. 43/61
Equation (57) with the help of (52)-(54) can be written as

\[
(1 - \sigma)c^{-\sigma}[\eta_t + (k^\beta - \delta k - c)\eta_k] - \eta(1 - \sigma)c^{-\sigma}(\beta k^{\beta - 1} - \delta) \\
- [c^{1-\sigma} + (1 - \sigma)c^{-\sigma}(k^\beta - \delta k - c)][\xi_t + (k^\beta - \delta k - c)\xi_k] \\
= B_t + (k^\beta - \delta k - c)B_k - r(1 - \sigma)c^{-\sigma}[\eta - \xi(k^\beta - \delta k - c)].
\]

(58)
Separating equation (58) with respect to powers of the control variable $c$, we have

\begin{align*}
c^{2-\sigma} & : -\sigma \xi_k = 0, \quad \text{(59)} \\
c^{1-\sigma} & : -\eta_k (1 - \sigma) - \sigma \xi_t + r (1 - \sigma) \xi = 0, \quad \text{(60)} \\
c^{-\sigma} & : \eta_t + (k^{\beta} - \delta k) \eta_k - \eta (\beta k^{\beta-1} - \delta) \\
& \quad - (k^{\beta} - \delta k) \xi_t + r \eta - r \xi (k^{\beta} - \delta k) = 0, \quad \text{(61)} \\
c, \ c^0 & : B_k = 0, \ B_t = 0. \quad \text{(62)}
\end{align*}
Equations (59), (60) and (62) results in

\[ \xi = a_1(t), \quad \eta = \left( -\frac{\sigma}{1 - \sigma} \dot{a}_1 + ra_1 \right)k + a_2(t), \quad B = 0. \tag{63} \]

Equation (61) with \( \xi, \eta, B \) from (63) gives \( a_2 = 0 \) and then reduces to

\[ k^\beta : \dot{a}_1 - \frac{\beta r(1 - \sigma)}{\beta \sigma - 1} a_1 = 0, \quad \beta \sigma \neq 1, \tag{64} \]

\[ k : -\sigma \ddot{a}_1 + \left[ r(1 - 2\sigma) + \delta(1 - \sigma) \right] \dot{a}_1 + r(1 - \sigma)(r + \delta)a_1 = 0. \tag{65} \]
Equation (64) is valid if $\sigma \beta \neq 1$, for the case where the capital’s share is not equal to the intertemporal elasticity of substitution. Equations (64) and (65) yield

$$a_1(t) = c_1 e^{\delta \beta (1-\sigma)t} \quad (66)$$

with

$$\sigma = \frac{r + \delta}{\beta \delta}. \quad (67)$$

The restriction on the parameters (67) is the same as given in Baro and Sala- i-Martin (2004); Ragni et al. (2010) and our approach yields this during the solution process.
Now $\xi, \eta$ and $B$ are given by

$$\xi = c_1 e^{\delta \beta (1-\sigma) t}, \quad \eta = -c_1 \delta e^{\delta \beta (1-\sigma) t} k, \quad B = 0,$$

and only partial Hamiltonian operator is

$$X = e^{\delta \beta (1-\sigma) t} \frac{\partial}{\partial t} - \delta e^{\delta \beta (1-\sigma) t} k \frac{\partial}{\partial k}, \quad B = 0.$$
The following first integral corresponding to the partial Hamiltonian operator and gauge terms given in (69) can be computed from (17):

\[ I = e^{\delta \beta (1-\sigma) t} \left[ -\sigma c^{1-\sigma} + (\sigma - 1) c^{-\sigma} k^\beta \right]. \]  

(70)

We write (70) as a constant, i.e.

\[ -\sigma c^{1-\sigma} + (\sigma - 1) c^{-\sigma} k^\beta = A_1 e^{\delta \beta (\sigma - 1) t}. \]  

(71)
From equation (71), we have

\[ k = \left[ \frac{A_1}{\sigma - 1} c^\sigma e^{\delta(\sigma - 1)t} + \frac{\sigma}{\sigma - 1} c \right]^\frac{1}{\beta}. \]  

(72)

Our next goal is to get either \( c \) or \( k \). If \( A_1 = 0 \) we arrive at the well-known solution given by Baro and Sala-i-Martin (2004); Ragni et al. (2010). Equation (72) for \( A_1 = 0 \) yields

\[ c(t) = (1 - \frac{\beta \delta}{r + \delta}) k^\beta \]  

(73)

where \( \frac{\sigma - 1}{\sigma} = 1 - \frac{\beta \delta}{r + \delta} \) by (67).
Substituting $c$ from equation (73) in Equation (53) results in

$$\dot{k} + \delta k = \left( \frac{\beta \delta}{r + \delta} \right) k^\beta.$$  \hspace{1cm} (74)

The solution of equation (74) subject to initial condition $k(0) = k_0$ is given by

$$k(t) = \left[ \frac{\beta}{r + \delta} + (k_0^{1-\beta} - \frac{\beta}{r + \delta}) e^{-\frac{(1-\beta)\delta t}{1-\beta}} \right]^\frac{1}{1-\beta}.$$ \hspace{1cm} (75)

The solutions (73) and (75) are the same as derived in and satisfy the transversality condition given by (55). This guarantees that our approach works. For $A_1 \neq 0$, we can get more solutions.
One-Sector Model of Endogenous growth: The AK model
We consider the following one-sector model of endogenous growth where the representative consumer’s consumer’s utility maximization problem is

$$\max \int_0^\infty e^{-(\rho-n)t} \frac{c^{1-\theta} - 1}{1-\theta} \, dt, \quad \theta > 0, \ \theta \neq 1$$

subject to

$$\dot{a}(t) = (r - n)a + w - c, \ c(0) = c_0,$$

where $c(t)$ is the consumption per person, $a(t)$ is the assets per person, $r(t)$ is the interest rate, $w(t)$ is the wage rate, and $n$ is the growth rate of population.
Suppose firms have the linear production function

$$y = f(k) = Ak$$

(78)

where $A > 0$. The marginal product of capital is not diminishing, i.e. $f'' = 0$ and this property makes it different from neoclassical production function. The marginal product of capital is the constant $A$ and marginal product of labor is zero. Thus

$$r = A - \delta, \ w = 0$$

(79)

where $\delta \geq 0$ is the depreciation rate.
It is assumed that the economy is closed and \( a(t) = k(t) \) holds. If we take \( a = k \), \( r = A - \delta \) and \( w = 0 \) then our optimal control problem is to maximize (76) subject to

\[
\dot{k} = (A - \delta - n)k - c, \quad c(0) = c_0. \tag{80}
\]

The current value Hamiltonian function is defined as

\[
H(t, c, A, \lambda) = \frac{c^{1-\theta} - 1}{1 - \theta} + \lambda[(A - \delta - n)k - c], \tag{81}
\]

where \( c(t) \) is control variable, \( k(t) \) is the state variable and \( \lambda(t) \) is the costate variable.
The necessary first order conditions for optimal control are

\[ \lambda = c^{-\theta} \]  \hspace{1cm} (82)

\[ \dot{k} = (A - \delta - n)k - c \]  \hspace{1cm} (83)

\[ \dot{\lambda} + (A - \delta - \rho)\lambda = 0. \]  \hspace{1cm} (84)

The transversality condition is

\[ \lim_{t \to \infty} e^{-(\rho-n)t} \lambda(t)k(t) = 0 \]  \hspace{1cm} (85)
From (82) and (84), the growth rate of consumption is given by

\[ \frac{\dot{c}}{c} = \frac{1}{\theta} (A - \delta - \rho). \quad (86) \]

We obtain following restriction on the parameters

\[ \rho + \delta > (1 - \theta)(A - \delta) + n\theta + \delta \quad (87) \]

The capital labor ratio \( k(t) \) is given by

\[ k(t) = \frac{1}{\phi} c_0 e^{(\frac{A - \delta - \rho}{\theta})t}, \quad (88) \]

with

\[ k(t) = \frac{1}{\phi} c(t). \quad (89) \]
Conclusions

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- This can be applied to many state and costate variables of the current value Hamiltonian.

- Two economic growth models, the Ramsey model with constant relative risk aversion (CRRA) utility function with Cobb Douglas technology and the one-sector AK model of endogenous growth were studied and the solutions derived from our methodology were the same as those derived in literature.
The restriction on the parameters was obtained in a systematic way during the solution process unlike in other approaches where it was assumed.

We have shown that our systematic approach can be used to deduce results given in the literature and we also found new solutions for a variety of models.
Future works

- Development of computer programm on Maple, Mathematica or Matlab for mathematical formulation presented
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- Formulation and solution of new models in Economic growth theory and other applications
Acknowledgements

We thank Professor Asghar Qadir, Dr Sajid Ali and the organizing committee of SDEAII for their kind invitation and making this talk possible. My gratitude also goes to Professor Rehana Naz for her invaluable help for my presentation. We are grateful to Wits and the NRF of South Africa for research funding.
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